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Electron on an arbitrary surface of revolution in a magnetic field

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Abstract. The energy spectrum of an electron confined to a mesoscopic surface of revolution in an external magnetic field, parallel to the symmetry axis, is studied analytically. Via conformal mapping the problem is reduced to the problem on the surface of a sphere. Cases of the sphere and the spheroid are considered in detail and the dependence on parameters is discussed. In the high magnetic field limit we observe a Landau level-like regular structure of the electron energy spectrum.

1. Introduction

The quantum mechanics of non-interacting electrons in a magnetic field is a rich subject both mathematically and physically. Initially attention was attracted to the problem of an electron in a parabolic potential [1] and on the infinite plane [2, 3]. Later the solutions for an electron in mesoscopic rings and cylinders [4, 5] were studied, motivating the observation [6] of the topologically nontrivial Aharonov–Bohm [7, 8] like effects. The electron spectrum in an oval-shaped stadium was studied in [9] and it was shown there that this model is relevant to the notion of chaos in the level statistics and related thermodynamics of such systems. The energy spectrum of the two-dimensional interacting electrons under a strong magnetic field was obtained in [10, 11], and generalized for the case when a one-dimensional periodic potential is applied in [12, 13].

Recently, there has been growing interest in electrons confined to a three-dimensional surface with a magnetic field applied along one of the symmetry axes. The case of a sphere was studied in [14, 15]. The energy spectrum was calculated there and the thermodynamic properties, such as magnetization and susceptibility were studied. Real systems rarely have a purely spherical shape, and it is desirable to know the electron spectrum for a surface of more general shapes. Here we consider an electron on an arbitrary surface of revolution placed in a uniform magnetic field. Our goal is to investigate the influence of the geometrical characteristics on the quantum-mechanical spectrum of the electron.

Consider the case of a single electron confined to the surface $r = f(z)$, where (r, φ, z) are the cylindrical coordinates. We assume the surface to be smooth, closed and to cross the z-axis only at two points $z = z_k$, $k = 1, 2$. The uniform magnetic field B points in the z-direction.

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The problem is described by the Hamiltonian

$$\mathcal{H} = \frac{1}{2m}\left[\mathrm{i}\hbar\nabla - \frac{e}{c}\boldsymbol{A}\right]^2 + V \tag{1}$$

where, for simplicity, we ignore spin-dependent terms. $\boldsymbol{A} = B(-y, x, 0)/2$ is the vector potential in the symmetric gauge [3], (x, y) are the Cartesian coordinates. This Hamiltonian leads to the following Schrödinger equation on the surface:

$$\left(\Delta + 2\mathrm{i}B_1\frac{\partial}{\partial\varphi} - B_1^2r^2 - V_1\right)\psi = -E_1\psi \tag{2}$$

where $E_1 = \left(2m/\hbar^2\right)E$, $B_1 = eB/2c\hbar$, $V_1 = \left(2m/\hbar^2\right)V$.

We introduce new orthogonal coordinates by $z + \mathrm{i}r = F(u + \mathrm{i}v)$, where the function $F(u + \mathrm{i}v)$ maps conformally the domain of the (u, v)-plane containing the unit circle onto the domain of the (r, z)-plane containing the closed curve $r = \pm f(z)$ and this curve is the image of the circle $u^2 + v^2 = 1$ with the arc $v \geqslant 0$ corresponding to $r \geqslant 0$.

In the new quasi-spherical polar coordinates $R\exp(\mathrm{i}\theta) = u + \mathrm{i}v$, equation (2) takes the following form:

$$\left[\frac{1}{R}\frac{\partial}{\partial R}R\frac{\partial}{\partial R} + \frac{1}{R^2}\frac{\partial^2}{\partial\theta^2} + \frac{1}{r}\left(\frac{\partial r}{\partial R}\frac{\partial}{\partial R} + \frac{1}{R^2}\frac{\partial r}{\partial\theta}\frac{\partial}{\partial\theta}\right)\right]\psi$$
$$= -|F'(R\exp(\mathrm{i}\theta)|^2\left[E_1 - V_1 - B_1^2r^2 + 2\mathrm{i}B_1\frac{\partial}{\partial\varphi} + \frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2}\right]\psi \tag{3}$$

where $r = \operatorname{Im} F(R\exp(\mathrm{i}\theta))$.

Since conformal mapping conserves a normal to the surface, we are allowed to write equation (3) on the surface $R = 1$ neglecting derivatives in R. Thus, the three-dimensional Schrödinger operator has been reduced to a two-dimensional operator in (θ, φ)-variables.

Due to the conservation of the z-component of the angular momentum, the cyclic coordinate φ can be separated in the Fourier series development

$$\psi(\theta, \varphi) = \sum_{m=-\infty}^{+\infty}\psi_m(\theta)\exp(\mathrm{i}m\varphi). \tag{4}$$

Further simplification $x = \cos\theta$ results in the ordinary differential equation of the second order

$$\left(1 - x^2\right)\frac{\mathrm{d}^2\psi_m}{\mathrm{d}x^2} - G_1(x)\frac{\mathrm{d}\psi_m}{\mathrm{d}x} + G_0(x)\,\psi_m = 0 \qquad |x| \leqslant 1 \qquad |\psi_m(\pm 1)| < \infty. \tag{5}$$

Here $G_1(x) = x - \left(1 - x^2\right)\rho'(x)\rho^{-1}(x)$, $\rho_0\rho(x) = \operatorname{Im} F\left(x + \mathrm{i}\sqrt{1 - x^2}\right)$, $\rho_0 = \max \operatorname{Im} F(\exp(\mathrm{i}\theta))$, $G_0(x) = \Phi(x)\left[\lambda - \tilde{B}^2\rho^2(x) - m^2\rho^{-2}(x)\right]\rho_0^{-2}$, $\Phi(x) = \left|F'\left(x + \mathrm{i}\sqrt{1 - x^2}\right)\right|^2$, $\lambda = \tilde{E} - 2\tilde{B}m$, $\tilde{E} = (E_1 - V_1)\rho_0^2$, $\tilde{B} = B_1\rho_0^2$.

The low field ($\tilde{B} \ll 1$) asymptotic behaviour of the spectrum and eigenfunctions can be found in the traditional way by the perturbation method. It is much more difficult to suggest a general approach to indicate a high-field ($\tilde{B} \gg 1$) asymptotic behaviour. This is governed by coefficients of equation (5) or, in other words, by the surface shape. These coefficients are continuous functions within the interval (1, 1) depending upon the harmonics label m and the geometrical parameter $\xi_0 = \left|\frac{1}{2\rho_0}(z_1 - z_2)\right|^2$. We will call a surface long as $\xi_0 \gg 1$ and flattened as $\xi_0 \ll 1$.

We consider some specific examples.

2. A spherical surface

In this case equation (5) is

$$\left(1-x^2\right)\frac{\mathrm{d}^2\psi_m}{\mathrm{d}x^2}-2x\frac{\mathrm{d}\psi_m}{\mathrm{d}x}+\left[\lambda-\tilde{B}^2\left(1-x^2\right)-\frac{m^2}{1-x^2}\right]\psi_m=0 \qquad (6)$$
$$|\psi_m(\pm 1)|<\infty \qquad |x|\leqslant 1$$

where ρ_0 is a radius of a sphere.

This is the well known equation for the oblate angular spheroidal functions [16]. Its eigenfunctions $\psi_{lm}(x)$ are even (odd) functions in x for even (odd) l. They correspond to a simple discrete spectrum and the eigenvalues are the roots of the transcendental equation

$$\beta_0-\frac{\alpha_0\gamma_1}{\beta_1-}\frac{\alpha_1\gamma_2}{\beta_2-\cdots}=0. \qquad (7)$$

In this continuous fraction

$$\alpha_s=2(n+1)(n+|m|+1)$$
$$\beta_s=(n+|m|)(n+|m|+1)+2\tilde{B}(2n+|m|+1)-\lambda$$
$$\gamma_s=2(n+|m|)\tilde{B}$$

where $n=2s+\sin^2(\pi l/2)$, $s=0,1\ldots.$

In a low-field limit the asymptotic expansion is

$$\tilde{E}_{lm}=(l+|m|)(l+|m|+1)+2\tilde{B}m+\mathrm{O}\left(\tilde{B}^2\right) \qquad l=0,1\ldots \qquad (8)$$

showing that the problem is asymptotically degenerate.

The leading terms of the high-field asymptotic expansion

$$\frac{1}{2\tilde{B}}\tilde{E}_{lm}=l+|m|+m+\cos^2(\pi l/2)+\mathrm{O}\left(\tilde{B}^{-1}\right) \qquad (9)$$

display a more complicated type of degeneracy which is analogous to Landau levels. Numerical calculations in [14, 15] show the relatively high efficiency of this formula. Landau levels resembling spectrum arise at about $\tilde{B}\approx 6$ and this tendency progresses with increasing field strength.

In the high-field limit the eigenfunctions became localized in the vicinity of the poles $x=\pm 1$ and are expressed by the associated Laguerre polynomials

$$\psi_{lm}(x)=\left(\frac{x}{|x|}\right)^l\left(1-x^2\right)^{|m|/2}\exp\left(-\tfrac{1}{2}\tilde{B}\left(1-x^2\right)\right)L_n^{(|m|)}\left(\tilde{B}\left(1-x^2\right)\right)+\mathrm{O}\left(\frac{1}{\tilde{B}}\right) \qquad (10)$$
$$x\notin(-\varepsilon,\varepsilon) \qquad 2n=l-\sin^2(\pi l/2).$$

We note that equation (8) gives two leading terms of the asymptotic expansion as $n\gg 1$.

3. A spheroidal surface

Let a surface be a spheroid whose equation is

$$\frac{z^2}{a^2}+\frac{r^2}{b^2}=1. \qquad (11)$$

Conformal mapping

$$z+\mathrm{i}r=\frac{a-b}{2\varpi}+(a+b)\frac{\varpi}{2} \qquad \varpi=R\exp(\mathrm{i}\theta) \qquad (12)$$

is a one-to-one mapping of the unit circle $R=1$ onto this ellipse of the (r,z)-plane.

Equation (5) can be written in the form

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(1-x^2\right)\frac{\mathrm{d}\psi_m}{\mathrm{d}x}+\left[\lambda-\tilde{B}^2\left(1-x^2\right)-\frac{m^2}{1-x^2}\right]\left[\xi\left(1-x^2\right)+1\right]\psi_m=0. \tag{13}$$

In this equation

$$|x|\leqslant 1 \qquad |\psi_m(\pm 1)|<\infty \qquad \xi=a^2b^{-2}-1=\xi_0-1.$$

According to the Sturm–Liouville theory for problems with singular end-points, this problem has an infinite discrete spectrum λ_{lm}. Its eigenfunctions $\psi_{lm}(x)$ have l zeros in the interval $(-1,1)$. It follows, that if l is even (odd) integer, then these functions are even (odd).

In order to calculate the spectrum we represent $\psi_{lm}(x)$ as

$$\psi_{lm}(x)=\mathrm{Re}\left[u_{lm}(x)\exp\left(\tfrac{1}{2}\mathrm{i}\tilde{B}x^2\sqrt{\xi}\,\right)\right] \tag{14}$$

where $u_{lm}(x)$ are eigenfunctions of the problem

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}x}\left(1-x^2\right)\frac{\mathrm{d}u_{lm}}{\mathrm{d}x}+2\mathrm{i}\beta x\left(1-x^2\right)\frac{\mathrm{d}u_{lm}}{\mathrm{d}x}+\left[\lambda_1+\mathrm{i}\beta+(\chi-3\mathrm{i}\beta)x^2-\frac{m^2}{1-x^2}\right]u_{lm}=0\\&|x|\leqslant 1 \qquad |u_{lm}(\pm 1)|<\infty\\&\lambda_1=\left(\lambda-\tilde{B}^2\right)(1+\xi)-m^2\xi \qquad \chi=\tilde{B}^2(1+\xi)-\lambda\xi \qquad \beta=\tilde{B}\sqrt{\xi}.\end{aligned} \tag{15}$$

The functions $u_{lm}(x)$ are found by expanding in the associated Legendre polynomials

$$u_{lm}(x)=\sum_{s=0}^{\infty}c_n P_{n+|m|}^{|m|}(x). \tag{16}$$

Here indices $n=2s+\sin^2(\pi l/2)$ are either even or odd integers corresponding to the symmetric and antisymmetric solutions, respectively.

Substituting this development into the equation (15) yields two recurrence relations (separately for even and odd integers n)

$$A_s c_{n-2}+J_s c_n+D_s c_{n+2}=0 \tag{17}$$

where

$$\begin{aligned}A_s&=\frac{n(n-1)}{4(n+|m|-1)^2-1}[\chi-\mathrm{i}\beta(2n+2|m|-1)]\\J_s&=\frac{\chi[2n(2|m|+n+1)+2|m|-1]}{(2n+2|m|+1)-4}-(n+|m|)(n+|m|+1)+\lambda_1\\D_s&=\frac{(n+2|m|+2)(n+2|m|+1)}{4(n+|m|+2)^2-1}[\chi+\mathrm{i}\beta(2|m|+2n+3)].\end{aligned}$$

The spectrum is determined by equating the infinite determinants of these equations to zero and is given, therefore, by the roots of the following continued fractions:

$$0=J_0-\frac{A_1D_0}{J_1-}\,\frac{A_2D_1}{J_2-\cdots}. \tag{18}$$

These continued fractions are real, since A_sD_{s-1} take real values.

The sufficient condition for the absence of the eigenfunctions is that a coefficient of the $\psi_m(x)$ in equation (13) is a non-positive function within $(-1,1)$. This condition leads to

$$\lambda_{lm}>\min_{0\leqslant y\leqslant 1}\left(\tilde{B}^2y+m^2y^{-1}\right)$$

or

$$\tilde{E}_{lm}>\begin{cases}2\tilde{B}(|m|+m) & \tilde{B}\geqslant|m|\\(\tilde{B}+m)^2 & \tilde{B}<|m|.\end{cases} \tag{19}$$

Thus, all eigenvalues λ_{lm} are positive. They are large as one of the conditions (a) $l \gg 1$, (b) $|m| \gg 1$, (c) $\tilde{B} \gg 1$ and $m > 0$ is fulfilled. Below we point out leading terms of the corresponding asymptotic series.

As $l \gg 1$, the spectrum can be obtained with methods of the papers [17]. Particularly, the leading term is given by

$$\tilde{E}_{lm} = \frac{\pi^2(2l+2|m|+1)^2}{16(1+\xi)\boldsymbol{E}^2\left(\sqrt{\xi/(1+\xi)}\right)} + 2\tilde{B}m + \mathrm{O}\left(\frac{1}{l}\right) \tag{20}$$

where $\boldsymbol{E}(x)$ is a complete elliptic integral of the second kind.

As $|m| \gg 1$, the asymptotic behaviour can be found with stretching the variable

$$\lambda_{lm} = \tilde{E}_{lm} - 2\tilde{B}m = m^2 + \frac{2l+1}{\sqrt{1+\xi}}|m| + \mathrm{O}(1). \tag{21}$$

Eigenfunctions are expressed by Hermite polynomials and they are localized in the equator zone

$$\psi_{lm}(x) = \exp\left(-x^2\sqrt{\lambda_{lm}}\right)H_l\left(x\lambda_{lm}^{1/4}\right) + \mathrm{O}\left(m^{-2}\right) \qquad x \in (-\varepsilon, \varepsilon). \tag{22}$$

The spectrum of the not very long surface ($\xi \ll \tilde{B}$) is given, in the high-field limit ($\tilde{B} \gg 1$), by an asymptotic formula

$$\begin{gathered}\lambda_{lm} = 2\nu\tilde{B}S + m^2\xi + \frac{\xi}{2S^2}\left(3\nu^2 - m^2 + 1\right) - \tfrac{1}{2}\left(\nu^2 - m^2 + 1\right) + \mathrm{O}\left(\frac{1}{\tilde{B}}\right)\\ S = \sqrt{1 + \frac{\left(\nu^2 - m^2\right)\xi^2}{\tilde{B}^2}} - \frac{\nu\xi}{\tilde{B}} \qquad \nu = l + |m| + \cos^2(\pi l/2).\end{gathered} \tag{23}$$

As follows from this expression, λ_{lm} are large in the states $m \leqslant 0$ as well. If $\nu|\xi| \ll \tilde{B}$, then

$$\tilde{E}_{lm} = 2N\tilde{B} - \tfrac{1}{2}(1+\xi)N(N-2m) + \tfrac{1}{2}(1-\xi) + \mathrm{O}\left(\frac{\nu}{\tilde{B}}\right) \qquad N = \nu + m. \tag{24}$$

The corresponding expression of the eigenfunctions is expressed by Laguerre polynomials

$$\begin{gathered}\psi_{lm}(x) = \left(\frac{x}{|x|}\right)^l\left(1-x^2\right)^{|m|/2}\exp\left(-\tfrac{1}{2}\left(1-x^2\right)\tilde{B}S\right)L_n^{|m|}\left(\left(1-x^2\right)\tilde{B}S\right) + \mathrm{O}\left(\frac{1}{\tilde{B}}\right)\\ n = \tfrac{1}{2}\left(l - \sin^2(\pi l/2)\right) \qquad x \notin (-\varepsilon, \varepsilon).\end{gathered} \tag{25}$$

Hence it appears that an energy spectrum resembling the Landau levels is formed in the high-field limit. Every energy level corresponds to two asymptotically degenerate bound states labelled $(2k, m)$ and $(2k+1, m)$. The levels with the same number N constitute a bunch of parallel equidistant straight lines. In a given N-bunch these lines are placed in the order of increasing m and the upper line is the one with $l = 0, 1$; $m = \frac{1}{2}(N-1)$. As a spheroid is flattened ($(\xi+1)N \ll 1$), there is an asymptotic coalescence of a bunch into a single line. Splitting of this Landau level is increasing with growing bunch numbers N as well as spheroid length ξ. Unlike the classical Landau problem for a given $\tilde{B}$ an electron has only a finite family of bunches. Their number rises as the field strength increases.

A disc of radius ρ_0 (a planar circular billiard) is a limiting case of a strongly flattened spheroid ($\xi \to -1$). In this limit, the values of the eigenfunctions on both sides of the disc ($x > 0$ and $x < 0$) are added and according to equation (25) the antisymmetric eigenfunctions are cancelled out. Taking into account $\rho_0^2\left(1-x^2\right) = r^2$, we obtain

$$\begin{gathered}E_1 - V_1 = 2B_1(2n + |m| + m + 1) + \frac{1}{\rho_0^2} + \mathrm{O}\left(\frac{1}{\tilde{B}\rho_0^2}\right)\\ \psi_{nm}(r) = r^{|m|}\exp\left(-\tfrac{1}{2}r^2B_1\right)L_n^{|m|}\left(r^2B_1\right) + \mathrm{O}\left(\frac{1}{\tilde{B}}\right) \qquad r < \rho_0 - \varepsilon.\end{gathered}$$

This turns into the Landau solution when $\rho_0 = \infty$. In the high-magnetic-field limit the energy spectrum of the electron on a disc coalesces into the straight lines similar to the Landau levels for an electron on a plane. The states which do not satisfy the restriction $\tilde{B} \gg n + |m|$ may break, however, such an ideal picture. This analytical conclusion confirms the results of numerical calculations by Nakamura and Thomas [9].

We may suggest that as $\tilde{B} \gg \nu(\xi_0 + 1)$, the electron states on an arbitrarily shaped convex surface have the asymptotic behaviour $\tilde{E}_{lm} = \tilde{B}\eta(l + \mu(m))$, where η depends on the surface shape in the close vicinity of the poles $x = \pm 1$. This hypothesis is based on a rather obvious assumption that in the high-magnetic-field limit the leading asymptotic term is predetermined by flatness of the surface in the vicinity of the rotation axis (where an electron is trapped). Therefore the requirement of convexness is, probably, too restricting.

For more detailed information on the spectrum of an electron, confined to a spheroid we proceed to treat it as a function of the parameters.

Let $\psi_{lm}(x)$ obey equation (13) and $\zeta_{l_1 m_1}(x)$ be an eigenfunction of the same equation but with the eigenvalue $\lambda_{l_1 m_1}$, $\tilde{B} = \tilde{B}_1$, $\xi = \xi_1$. As a starting point we use the identity

$$\begin{aligned}(\xi - \xi_1) \int_{-1}^{1} &\left[\lambda_{l_1 m_1} - \tilde{B}_1^2\left(1 - x^2\right) - \frac{m^2}{1 - x^2}\right]\psi_{lm}(x)\, \zeta_{l_1 m_1}(x)\left(1 - x^2\right) \mathrm{d}x \\ &+ \int_{-1}^{1} \left[\Delta\lambda - \Delta\tilde{B}\left(1 - x^2\right) - \frac{\Delta m}{1 - x^2}\right]\psi_{lm}(x)\, \zeta_{l_1 m_1}(x)\left[1 + \xi\left(1 - x^2\right)\right] \mathrm{d}x \\ &= -\int_{-1}^{1} \mathrm{d}\left[\left(1 - x^2\right)\left(\zeta'_{l_1 m_1}(x)\, \psi_{lm}(x) - \zeta_{l_1 m_1}(x)\, \psi'_{lm}(x)\right)\right] = 0\end{aligned}$$

where $\Delta\lambda = \lambda_{l_1 m_1} - \lambda_{lm}$, $\Delta\tilde{B} = \tilde{B}_1^2 - \tilde{B}^2$, $\Delta m = m_1^2 - m^2$.

Let us now work out in detail this identity for various relations of the parameters.

(a) $\tilde{B}_1 = \tilde{B}$, $\xi_1 = \xi$, $m_1 = m$, $l_1 \neq l$. We obtain the orthogonality condition

$$\int_{-1}^{1} \left[1 + \xi\left(1 - x^2\right)\right] \psi_{lm}(x)\, \psi_{l_1 m}(x)\, \mathrm{d}x = 0. \tag{26}$$

(b) $\tilde{B}_1 = \tilde{B}$, $\xi = \xi_1$, $m_1 \neq m$, $\tilde{E}_{l_1 m_1} = \tilde{E}_{lm}$. The identity yields the degeneracy condition

$$\int_{-1}^{1} \left(\frac{m_1 + m}{1 - x^2} + 2\tilde{B}\right)\psi_{lm}(x)\, \psi_{l_1 m_1}(x)\left[1 + \xi\left(1 - x^2\right)\right] \mathrm{d}x = 0. \tag{27}$$

(c) $\xi_1 = \xi$, $m_1 = m$, $l_1 = l$, $\tilde{B}_1 \to \tilde{B}$. As a result of dividing by $\Delta\tilde{B}$, we have in the limit the following ordinary differential equation characterizing dependence of the spectrum upon the magnetic field:

$$\|\psi_{lm}(x)\|^2 \frac{\mathrm{d}\lambda_{lm}}{\mathrm{d}(\tilde{B}^2)} = \int_{-1}^{1} \left(1 - x^2\right) \psi_{lm}^2(x)\left[1 + \xi\left(1 - x^2\right)\right] \mathrm{d}x \tag{28}$$

$$\|\psi_{lm}(x)\|^2 \frac{\mathrm{d}\tilde{E}_{lm}}{\mathrm{d}\tilde{B}} = 2\int_{-1}^{1} \left[\tilde{B}\left(1 - x^2\right) + m\right] \psi_{lm}^2(x)\left[1 + \xi\left(1 - x^2\right)\right] \mathrm{d}x \tag{29}$$

where

$$\|\psi_{lm}(x)\| = \left[\int_{-1}^{1} \left[1 + \xi\left(1 - x^2\right)\right]\psi_{lm}^2(x)\, \mathrm{d}x\right]^{1/2}$$

is the norm of the eigenfunction $\psi_{lm}(x)$.

It follows from these equations

$$\begin{aligned}&2m(\tilde{B} - \tilde{B}_1) < \tilde{E}_{lm}(\tilde{B}) - \tilde{E}_{lm}(\tilde{B}_1) < (\tilde{B} - \tilde{B}_1)(\tilde{B} + \tilde{B}_1 + 2m) \\ &\tilde{B}_1^2\tilde{E}_{lm}(\tilde{B}) - \tilde{B}^2\tilde{E}_{lm}(\tilde{B}_1) < 2m\tilde{B}\tilde{B}_1(\tilde{B}_1 - \tilde{B}).\end{aligned}$$

The eigenvalues λ_{lm} are monotonically increasing functions of $\tilde{B}$. The energy levels $\tilde{E}_{lm}$ are monotonically increasing functions of $\tilde{B}$ when $m \geqslant 0$. As $m < 0$, the energy levels are monotonically decreasing in the interval $0 \leqslant \tilde{B} \leqslant -m$ and are increasing provided $\tilde{B} \gg -m$. They have extrema only in some fairly wide zone $(-m, \tilde{B}_0)$.

The obtained differential equations can be considered as relationships determining the average squared deviation of the electron from the equator $x = 0$ in the (l, m)-bound state

$$\bar{x}^2 = 1 - \frac{\mathrm{d}\lambda_{lm}}{\mathrm{d}\big(\tilde{B}^2\big)} = 1 + \frac{m}{\tilde{B}} - \frac{1}{2\tilde{B}}\frac{\mathrm{d}\tilde{E}_{lm}}{\mathrm{d}\tilde{B}}. \tag{30}$$

The average electron position is thus easily found by differentiating the dispersion relations.

(d) $\tilde{B}_1 = \tilde{B}, \xi_1 = \xi, l_1 = l, m_1 \to m$. We admit here that the parameter m is an arbitrary real number and derive the ordinary differential equations

$$\|\psi_{lm}(x)\|^2 \frac{\mathrm{d}\lambda_{lm}}{\mathrm{d}\big(m^2\big)} = \int_{-1}^{1} \frac{1 + \xi\big(1 - x^2\big)}{1 - x^2}\psi_{lm}^2(x)\,\mathrm{d}x \tag{31}$$

$$\|\psi_{lm}(x)\|^2 \frac{\mathrm{d}\tilde{E}_{lm}}{\mathrm{d}m} = 2\int_{-1}^{1} \frac{\tilde{B}\big(1 - x^2\big) + m}{1 - x^2}\psi_{lm}^2(x)\big[1 + \xi\big(1 - x^2\big)\big]\,\mathrm{d}x. \tag{32}$$

One can see that eigenvalues monotonically increase in $|m|$ and

$$0 < \lambda_{lm} - \lambda_{lm_1} < m^2 - m_1^2 \qquad \frac{\lambda_{lm}}{\lambda_{lm_1}} < \frac{m^2}{m_1^2} \qquad (|m| > |m_1| \neq 0). \tag{33}$$

Energy levels $\tilde{E}_{lm}$ constitute under magnetic field $\tilde{B}$ a monotonically increasing sequence as $m \geqslant 0$. When $m < 0$, their behaviour is much more intricate. They are only known to be monotonically decreasing as $m \leqslant -\tilde{B}$ or else as $l \gg -m, \tilde{B}$.

(e) $\tilde{B}_1 = \tilde{B}, m_1 = m, l_1 = l, \xi_1 \to \xi$. In this limit we obtain the ordinary differential equation showing influence of the spheroid geometry parameter on the quantum-mechanical spectrum

$$\|\psi_{lm}(x)\|^2 \frac{\mathrm{d}\tilde{E}_{lm}}{\mathrm{d}\xi} = -\int_{-1}^{1}\left[\tilde{E}_{lm} - \left(\tilde{B}\sqrt{1 - x^2} + \frac{m}{\sqrt{1 - x^2}}\right)^2\right]\psi_{lm}^2(x)\big(1 - x^2\big)\,\mathrm{d}x. \tag{34}$$

We observe a lowering of the energy levels as the spheroid becomes longer.

The case of a 'long' spheroid can be treated by asymptotic methods. For example, as $\sqrt{\xi} \gg \tilde{B} + |m| + 1$, leading terms of the corresponding asymptotic expansion have the structure

$$\tilde{E}_{lm} = (\tilde{B} + m)^2 + \frac{w(\tilde{B}, m, l)}{\sqrt{\xi}} + \mathrm{O}\left(\frac{1}{\xi}\right) \qquad w(\tilde{B}, m, l) > 0. \tag{35}$$

Under the same m two successive energy levels nearly do not differ. This condensing is a process of a continuous spectrum arising occupying the region $\tilde{E} \geqslant (\tilde{B} + m)^2$. As one might expect, such a limiting continuous spectrum is the spectrum of an infinitely long cylinder. The mentioned condensing is absent in the states with large labels m because equation (21) remains valid.

To conclude we have studied analytically the energy spectrum of an electron confined to an arbitrary surface of revolution in an external magnetic field, parallel to the symmetry axis. Via conformal mapping the problem is reduced to the problem on the surface of a sphere. The cases of a sphere and a spheroid are considered in detail and the dependence on parameters is discussed. In particular, a Landau-levels-like regular structure of the energy spectrum is observed in a high-magnetic-field limit. Every energy level corresponds to two asymptotically degenerate bound states. The levels with the same number constitute a bunch of parallel straight lines. The conditions are found when there is an asymptotic coalescence of a bunch into a single line. Unlike the classical Landau levels problem an electron has only a finite family of bunches. Their number rises as the field strength increases.

The dependence of the eigenfunctions and eigenvalues on the harmonics number is studied. We have obtained that the wavefunctions, corresponding to high harmonics are localized in the equator zone. In the high-field limit, on the other hand, they are concentrated near the poles.

Solutions of the Landau problem for a plane, problems of a planar circular billiard and an infinite circular cylinder are obtained as limits.

Our analytical results confirm the numerical calculations existing in the literature. With some modifications these results could be easily generalized for the anisotropic (orthotropic) case.

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